The Three Valued Semantics of Phenesthe+

Manolis Pitsikalis

May 2023

1 Three valued semantics

In this report, we define the three valued semantics of Phenesthe+ ($TPhL$), the extension of Phenesthe [2] ($TPhL_o$). While the original version of Phenesthe, allowed future formulae—i.e., their evaluation at an instant t depends on information after t—for dynamic temporal phenomena, it didn't allowed future formulae, for events and states. Phenesthe+ allows the definition of future formulae for all phenomena categories.

A *stream*, a *stream*, at any instant t can be represented by the finite model $\mathcal{M}_t = \langle T_t, I_t, <, V^{\bullet}, V^{-}, V^{=}\rangle$ where $T_t = \{0, 1, \cdots, t\}$, $I_t = T_t \times T_t \cup \{[ts, \infty) : ts \in T_t\}$, and $V^{\bullet} : \Phi^{\cdot}_o \to 2^T_t$, $V^{-} : \Phi^{-}_o \to 2^I_t$, $V^{=} : \Phi^{=}_o \to 2^I_t$ are valuations of atomic formulae.

A *stream processor*, in symbols $\mathcal{SP}_t$, is defined by the triplet $\langle \Lambda^{\bullet}_t, \Lambda^{-}_t, \Lambda^{=}_t\rangle$ where $t \in T$, $\Lambda^{\bullet}_t : \Phi^{\bullet} \times T_t \to \{\top, \bot, ?\}$, $\Lambda^{-}_t : \Phi^{-} \times (I^c_t \cup I^+_T) \to \{\top, \bot, ?\}$, and $\Lambda^{=}_t : \Phi^{-} \times I^c_t \cup I^+_t) \to \{\top, \bot, ?\}$, are formulae valuation functions assigning truth values on formulae-instants/intervals pairs and $I^c_t = T_t \times T_t$ and $I^+_t = \{[ts, t+] : ts, t \in T_t\}$. We assume that all input phenomena are true and ordered. Where appropriate we will use the connectives $\wedge^u, \vee^u$ and $\neg^u$ of Kleene's strong logic of indeterminacy [1].

1.1 Formulae of $\Phi^{\bullet}$

The semantics for formulae of $\Phi^{\bullet}$ using $\Lambda^{\bullet}_{t_q} : \Phi^{\bullet} \times T \to \top, \bot, ?$ are as follows:

- $\Lambda^{\bullet}_{t_q}(P(a_1, ..., a_n), t) = \begin{cases} \top & t \in V^{\bullet}(P(a_1, ..., a_n)) \\ \bot & t \notin V^{\bullet}(P(a_1, ..., a_n)) \\ ? & \text{never.} \end{cases}$ where P is an n-ary event predicate symbol.
- $\Lambda^{\bullet}_{t_q}(\neg\phi, t) = \neg^u \Lambda^{\bullet}_{t_q}(\phi, t)$ where $\phi \in \Phi^{\bullet}$.
- $\Lambda^{\bullet}_{t_q}(\phi \wedge \psi, t) = \Lambda^{\bullet}_{t_q}(\phi, t) \wedge^u \Lambda^{\bullet}_{t_q}(\psi, t)$, where $\phi, \psi \in \Phi^{\bullet}$.
- $\Lambda^{\bullet}_{t_q}(\phi \vee \psi, t) = \Lambda^{\bullet}_{t_q}(\phi, t) \vee^u \Lambda^{\bullet}_{t_q}(\psi, t)$, where $\phi, \psi \in \Phi^{\bullet}$.
- $\Lambda^{\bullet}_{t_q}(\mathsf{start}(\phi), t) = \begin{cases} \top, \bot & \exists te. \Lambda^{-}_{t_q}(\phi, [t, te]) = \top, \bot \\ ? & \exists_{\leq t} ts. \exists^{>t} te.\ \Lambda^{-}_{t_q}(\phi, [ts, te]) = ? \end{cases}$

  where $\phi \in \Phi^{-}$ and $\Lambda^{-}_{t_q}(\phi, [t, te])$ denotes denotes the valuation of formula $\phi \in \Phi^{-}$ at an interval $[t, te]$ as defined below.
- $\Lambda^{\bullet}_{t_q}(\mathsf{end}(\phi), t) = \begin{cases} \top, \bot & \exists ts. \Lambda^{-}_{t_q}(\phi, [ts, t]) = \top, \bot \\ ? & \exists_{<t} ts. \exists^{\geq t} te.\ \Lambda^{-}_{t_q}(\phi, [ts, te]) \ = ? \end{cases}$

  where $\phi \in \Phi^{-}$.
- $\Lambda^{\bullet}_{t_q}(\phi \in \psi, t) = \exists^{\leq t} ts. \exists_{\geq t} te.\ \Lambda^{\bullet}_{t_q}(\phi, t) \wedge^u \Lambda^{-}_{t_q}(\psi, [ts, te])$.

1.2 Formulae of $\phi \in \Phi^{-}$

The semantics for formulae $\phi \in \Phi^{-}$ given by means of the valuation function $\Lambda^{-}_{t_q} : \Phi^{-} \times I \to \{\top, \bot, ?\}$ are defined as follows:

- $\Lambda^{-}_{t_q}(P(a_1, ..., a_n), [ts, te]) = \begin{cases} \top & [ts, te] \in V^{-}(P(a_1, ..., a_n)) \\ \bot & [ts, te] \notin V^{-}(P(a_1, ..., a_n)) \\ ? & \text{never.} \end{cases}$

  where P is an n-ary state predicate symbol.

- $$\Lambda^{-}_{t_q}(\phi \rightarrowtail \psi, [ts, te]) = \begin{cases} \top & \Lambda^{\bullet}_{t_q}(\phi, ts) = \top \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) = \top \\ & \wedge\ \forall^{<te}_{>ts} t. \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) = \bot \\ & \wedge\ \forall^{<ts} ts'. [\Lambda^{\bullet}_{t_q}(\phi, ts') \neq \bot \rightarrow \\ & \quad \exists^{\leq ts}_{>ts'} te'. \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te') = \top] \\ \bot & \text{otherwise.} \\ ? & \Big[\Big[\big[\Lambda^{\bullet}_{t_q}(\phi, ts) = ? \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) = \top\big] \\ & \quad \vee \big[\Lambda^{\bullet}_{t_q}(\phi, ts) = \top \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) = \top \\ & \quad\quad \wedge\ \exists^{<te}_{>ts} t.\ \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) = ?\big]\Big] \\ & \quad \wedge\ \big[\forall^{<te}_{>ts} t. \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) \neq \top\big]\Big] \\ & \wedge\ \forall^{<ts} ts'. [\Lambda^{\bullet}_{t_q}(\phi, t') \neq \bot \rightarrow \\ & \quad \exists^{\leq ts}_{>ts'} te'. \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) = \top] \end{cases}$$

where $\phi, \psi \in \Phi^{-}$. Essentially, $\phi \rightarrowtail \psi$ holds true for the disjoint maximal intervals that start at the earliest instant ts where ϕ is true and end at the earliest instant $te, te > ts$ where ψ is true and ϕ is false, while in the meantime there is no instant te' where $\psi \wedge \neg\phi$ is unknown (?). If on the other hand, there is an instant te' where $\psi \wedge \neg\phi$ is unknown between ts and te where ϕ and $\psi \wedge \neg\phi$ are true or if at the earliest instant ts ϕ is unknown before the earliest instant te where $\psi \wedge \neg\phi$ is unknown then $\phi \rightarrowtail \psi$ holds unknown for the interval $[ts, te]$.

- $$\Lambda^{-}_{t_q}(\phi \rightarrowtail \psi, [ts, tq+]) = \begin{cases} \top & \Lambda^{\bullet}_{t_q}(\phi, ts) = \top \wedge \forall_{>ts} t\ \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) = \bot\ \wedge \\ & \forall^{<ts} ts'. [\Lambda^{\bullet}_{t_q}(\phi, t') \neq \bot \rightarrow \exists^{\leq ts}_{>ts'} te'. \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) = \top] \\ \bot & \text{otherwise.} \\ ? & \big[\Lambda^{\bullet}_{t_q}(\phi, ts) = ? \vee \big[\Lambda^{\bullet}_{t_q}(\phi, ts) = \top \\ & \quad \wedge\ \exists_{>ts} t.\ \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) = ?\big]\big] \\ & \wedge\ \forall_{>ts} t.\ \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) \neq \top \\ & \wedge\ \forall^{<ts} ts'. [\Lambda^{\bullet}_{t_q}(\phi, ts') \neq \bot \rightarrow \\ & \quad \exists^{\leq ts}_{>ts'} te'. \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te') = \top] \end{cases}$$

In this case, the maximal range operator holds for an interval with a known start but unknown end. For true valuations, this means that a state has started being true, and will continue to be true up to at least tq. For a valuation with unknown status it means that the formula might hold for an interval starting in the range $[ts, tq+]$.

- $$\Lambda^{-}_{t_q}(\phi \looparrowright \psi, [ts, te]) = \begin{cases} \top & \Lambda^{\bullet}_{t_q}(\phi, ts) = \top \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) = \top \\ & \wedge\ \forall^{<te}_{>ts} t. \big[\Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) = \bot\ \wedge \Lambda^{\bullet}_{t_q}(\phi, t) = \bot\ \big] \\ \bot & \text{otherwise.} \\ ? & \Big[\Lambda^{\bullet}_{t_q}(\phi, ts) = \top \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) = \top \\ & \quad \wedge\ \forall^{<te}_{>ts} t. \big[\Lambda^{\bullet}_{t_q}(\phi, t) \neq \top \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) \neq \top\big] \\ & \quad \wedge\ \exists^{<te}_{>ts} t. \big[\Lambda^{\bullet}_{t_q}(\phi, t) = ? \vee \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) = ?\big]\Big] \\ & \vee\ \Big[\Lambda^{\bullet}_{t_q}(\phi, ts) = ? \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) = \top \\ & \quad \wedge\ \forall^{<te}_{>ts} t \big[\Lambda^{\bullet}_{t_q}(\phi, t) \neq \top \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) \neq \top\big] \\ & \quad \wedge\ \forall^{<ts} t.\ \Lambda^{\bullet}_{t_q}(\phi, t) \neq \bot \rightarrow \exists^{\leq ts}_{>t} t' \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t') \neq \bot\Big] \\ & \vee\ \Big[\Lambda^{\bullet}_{t_q}(\phi, ts) \neq \bot \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, te) = ? \\ & \quad \wedge\ \forall^{<te}_{>ts} t \big[\Lambda^{\bullet}_{t_q}(\phi, t) \neq \top \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) \neq \top\big] \\ & \quad \wedge\ \forall^{<ts} t.\ \Lambda^{\bullet}_{t_q}(\phi, t) \neq \bot \rightarrow \exists^{\leq ts}_{>t} t' \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t') \neq \bot \\ & \quad \wedge\ \exists_{>te} ts'. \big[\Lambda^{\bullet}_{t_q}(\phi, ts') = \top \wedge \forall^{\leq ts'}_{>te} t. \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) = \bot\big]\Big] \end{cases}$$

Here a formula $\phi \looparrowright \psi$ is true for an interval $[ts, te]$ if $\psi \wedge \neg\phi$ is true at te and ϕ is true at the latest instant ts before te, while in the meantime there are no instants t' where either ϕ or $\psi \wedge \neg\phi$ are unknown. $\phi \looparrowright \psi$ is unknown if it falls in any of the categories depicted in Figure 1. In cases (a) and (b) the interval starts at an instant where ϕ is either true or unknown, and ends at an instant where $\psi \wedge \neg\phi$ is true, while in the meantime ϕ or $\psi \wedge \neg\phi$ are unknown at least once. In case (c) the interval starts at an instant where ϕ is unknown and ends at an instant where $\psi \wedge \neg\phi$ is true, while in the meantime both ϕ

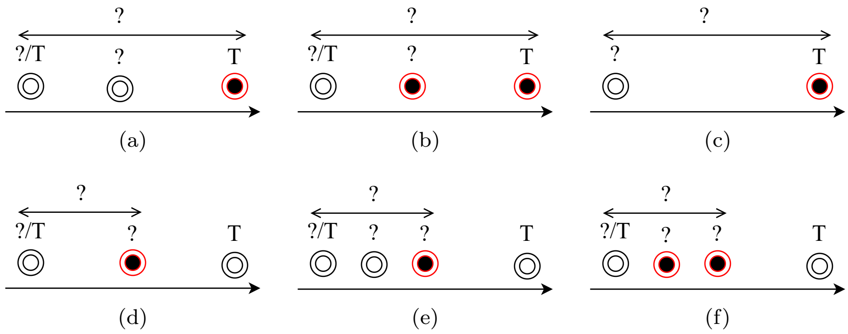

Figure 1: Valuation cases of $\phi \looparrowright \psi$ where the truth value is unknown. Circles with an empty centre denote the valuation of the formula ϕ, while red circles with a filled centre denote the valuation of the formula $\psi \wedge \neg\phi$. Above the circles is the valuation result of each formula, while the double arrow line denotes the interval with the unknown truth value.

and $\psi \wedge \neg\phi$ are false. Cases (d), (e) and (f) are similar to the previous ones, however with the difference that the interval ends at instant where $\psi \wedge \neg\phi$ is unknown, and directly after ϕ is true.

$$\Lambda^{-}_{t_q}(\phi \looparrowright \psi, [ts, tq+]) = \begin{cases} \top & \text{never.} \\ \bot & \text{otherwise.} \\ ? & \Big[\Lambda^{\bullet}_{t_q}(\phi, ts) = \top \\ & \quad \wedge \forall_{>ts} t. \big[\Lambda^{\bullet}_{t_q}(\phi, t) \neq \top \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) \neq \top\big]\Big] \\ & \vee \Big[\Lambda^{\bullet}_{t_q}(\phi, ts) = ? \\ & \quad \wedge \forall_{>ts} t. \big[\Lambda^{\bullet}_{t_q}(\phi, t) \neq \top \wedge \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t) \neq \top\big] \\ & \wedge \forall^{<ts} t.\ \Lambda^{\bullet}_{t_q}(\phi, t) \neq \bot \rightarrow \exists^{\leq ts}_{>t} t' \Lambda^{\bullet}_{t_q}(\psi \wedge \neg\phi, t') \neq \bot\Big] \end{cases}$$

- For the semantics of temporal union first we define sequences of overlapping intervals. A sequence of k ($k > 0$) overlapping intervals $i^o_j = [ts^o_j, te^o_j]$ is denoted by formulae $EI^k_o(...)$ which are defined as follows:

$$EI^k_o = \exists i^o_1. \ ...\ \exists i^o_k.\ \big[\forall^{<k}_{\geq 1} j.\ te^o_j \in i^o_{j+1} \wedge ts^o_j < ts^o_{j+1} \wedge te^o_j < te^o_{j+1} \wedge (...)\big] \qquad (k > 1)$$

$$EI^k_o = \exists i^o_1.\ (...) \qquad (k = 1)$$

Below we define the three valued semantics[1] of temporal union for $k > 0$.

[1] A first order definition of the semantics is possible, but the formula is not intuitive and lengthy.

$$\Lambda^-_{t_q}(\phi \sqcup \psi, [ts, te]) = \begin{cases} \top & EI^k_o \Big[ts^o_1 = ts \wedge te^o_k = te \wedge \forall^{\leq k}_{>0} j.\ \big[\Lambda^-_{t_q}(\phi, i^o_j) = \top \\ & \quad \vee\ \Lambda^-_{t_q}(\psi, i^o_j) = \top\big]\Big] \\ & \wedge\ \forall^{\leq ts} ts'.\forall_{\geq te} te'.\ \Big[[ts, te] \neq [ts', te'] \rightarrow \\ & \quad \neg\Big[EI^z_\mu.\ \big[ts^\mu_1 = ts' \wedge te^\mu_z = te' \\ & \qquad \wedge\ \forall^{\leq z}_{>1} j.\ \big[\Lambda^-_{t_q}(\phi, i^\mu_j) \neq \bot \vee\ \Lambda^-_{t_q}(\psi, i^\mu_j) \neq \bot\big]\big]\Big]\Big] \\ \bot & \text{otherwise.} \\ ? & EI^k_o \Big[ts^o_1 = ts \wedge te^o_k = te \wedge \forall i \in I^k_o.\ \big[\Lambda^-_{t_q}(\phi, i^o) \neq \bot \\ & \quad \vee\ \Lambda^-_{t_q}(\psi, i^o) \neq \bot\big] \\ & \quad \wedge\ \exists^{\leq k}_{>0} j.\ \Big[\big[\Lambda^-_{t_q}(\phi, i^o) = ?\ \wedge\ \Lambda^-_{t_q}(\psi, i^o) = \bot\big] \\ & \qquad \vee\ \big[\Lambda^-_{t_q}(\phi, i^o) = \bot \wedge\ \Lambda^-_{t_q}(\psi, i^o) = \bot\big]\Big]\Big] \\ & \wedge\ \forall^{\leq ts} ts'.\forall_{\geq te} te'. \Big[[ts, te] \neq [ts', te'] \rightarrow \\ & \quad \neg\Big[EI^z_\mu\ \big[ts^\mu_1 = ts' \wedge te^\mu_z = te' \\ & \qquad \wedge\ \forall^{\leq z}_{>1} j.\ \big[\Lambda^-_{t_q}(\phi, i_j) \neq \bot \vee\ \Lambda^-_{t_q}(\psi, i_j) \neq \bot\big]\big]\Big]\Big] \end{cases}$$

where ϕ, ψ are formulae of Φ^-. In simple terms, the temporal union $\phi \sqcup \psi$ holds true for the maximal intervals that are made up from a sequence of intervals where at least one of ϕ or ψ holds true. The temporal union $\phi \sqcup \psi$ holds for a maximal interval with unknown truth value if in there is at least one interval in EI^k_o at which ϕ or ψ exclusively holds unknown.

- $$\Lambda^-_{t_q}(\phi \sqcap \psi, [ts, te]) = \begin{cases} \top & \forall^{\leq te}_{\geq ts} t.\exists^{\leq t} ts'_\phi.\exists^{\leq t} ts'_\psi.\exists_{\geq t} te'_\phi.\exists_{\geq t} te'_\psi. \\ & \big[\Lambda^-_{t_q}(\phi, [ts'_\phi, te'_\phi]) = \top \wedge\ \Lambda^-_{t_q}(\psi, [ts'_\psi, te'_\psi]) = \top\big] \\ & \wedge \neg\Big[\exists^{\leq ts} ts'.\exists_{\geq te} te'.[ts', te'] \neq [ts, te] \\ & \quad \wedge\Big[\forall^{\leq te'}_{\geq ts'} t.\exists^{\leq t} ts''_\phi.\exists^{\leq t} ts''_\psi.\exists_{\geq t} te''_\phi.\exists_{\geq t} te''_\psi. \\ & \qquad \big[\Lambda^-_{t_q}(\phi, [ts''_\phi, te''_\phi]) \neq \top \\ & \qquad\quad \wedge\ \Lambda^-_{t_q}(\psi, [ts''_\psi, te''_\psi]) \neq \top\big]\Big]\Big] \\ \bot & \text{otherwise.} \\ ? & \forall^{\leq te}_{\geq ts} t.\exists^{\leq t} ts'_\phi.\exists^{\leq t} ts'_\psi.\exists_{\geq t} te'_\phi.\exists_{\geq t} te'_\psi. \\ & \Big[\big[\Lambda^-_{t_q}(\phi, [ts'_\phi, te'_\phi]) = \top \wedge\ \Lambda^-_{t_q}(\psi, [ts'_\psi, te'_\psi]) = ?\big] \\ & \quad \vee\big[\Lambda^-_{t_q}(\phi, [ts'_\phi, te'_\phi]) = ? \wedge\ \Lambda^-_{t_q}(\psi, [ts'_\psi, te'_\psi]) = \top\big] \\ & \quad \vee\big[\Lambda^-_{t_q}(\phi, [ts'_\phi, te'_\phi]) = ? \wedge\ \Lambda^-_{t_q}(\psi, [ts'_\psi, te'_\psi]) = ?\big]\Big] \\ & \wedge \neg\Big[\exists^{\leq ts} ts'.\exists_{\geq te} te'.[ts', te'] \neq [ts, te] \\ & \quad \wedge\Big[\forall^{\leq te'}_{\geq ts'} t.\exists^{\leq t} ts''_\phi.\exists^{\leq t} ts''_\psi.\exists_{\geq t} te''_\phi.\exists_{\geq t} te''_\psi. \\ & \qquad \big[\Lambda^-_{t_q}(\phi, [ts''_\phi, te''_\phi]) \neq \top \\ & \qquad\quad \wedge\ \Lambda^-_{t_q}(\psi, [ts''_\psi, te''_\psi]) \neq \top\big]\Big]\Big] \end{cases}$$

where $\phi, \psi \in \Phi^-$. In other words, the temporal intersection of two formulae of Φ^- holds true for the maximal sub-intervals of the intervals where both formulae hold true. The temporal intersection holds unknown for an interval if both formulae hold for intervals with unknown status, or one holds for an interval with unknown status and the other holds true.

- $$\Lambda^-_{t_q}(\phi \ \setminus \ \psi, [ts, te]) = \begin{cases} \top & \forall^{\leq te}_{\geq ts} t. \big[\exists^{\leq t} ts'. \exists_{\geq t} te'. \Lambda^-_{t_q}(\phi, [ts'_\phi, te'_\phi]) = \top \wedge \ \forall^{\leq t} ts'. \forall_{\geq t} te'. \\ & \Lambda^-_{t_q}(\psi, [ts', te']) = \bot\big] \wedge \neg \Big[\exists^{\leq ts} ts'. \exists_{\geq te} te'. \ [ts', te'] \neq [ts, te] \\ & \wedge \ \forall^{\leq te'}_{\geq ts'} t. \big[\exists^{\leq t} ts''. \exists_{\geq t} te''. \Lambda^-_{t_q}(\phi, [ts'', te'']) \neq \bot \\ & \wedge \ \forall^{\leq t} ts''. \forall_{\geq t} te''. \wedge \Lambda^-_{t_q}(\psi, [ts'', te'']) \neq \top\big]\Big] \\ \bot & \text{otherwise.} \\ ? & \forall^{\leq te}_{\geq ts} t. \ \Big[\big[\exists^{\leq t} ts'. \exists_{\geq t} te'. \Lambda^-_{t_q}(\phi, [ts', te']) = \ ? \wedge \ \forall^{\leq t} ts'. \forall_{\geq t} te'. \\ & \Lambda^-_{t_q}(\psi, [ts', te']) \neq \top\big] \vee \ \Big[\big[\exists^{\leq t} ts'. \exists_{\geq t} te'. \Lambda^-_{t_q}(\phi, [ts', te']) = \top \\ & \wedge \ \forall^{\leq t} ts'. \forall_{\geq t} te'. \Lambda^-_{t_q}(\psi, [ts', te']) \neq \top\big] \wedge \ \exists^{\leq t} ts'. \exists_{\geq t} te'. \\ & \Lambda^-_{t_q}(\psi, [ts', te']) = ?\Big]\Big] \wedge \neg \Big[\exists^{\leq ts} ts'. \exists_{\geq te} te'. \ \Big[[ts', te'] \neq [ts, te] \\ & \wedge \forall^{\leq te'}_{\geq ts'} t. \ \big[\exists^{\leq t} ts''. \exists_{\geq t} te''. \Lambda^-_{t_q}(\phi, [ts'', te'']) \neq \bot \wedge \ \forall^{\leq t} ts'. \forall_{\geq t} te'. \\ & \Lambda^-_{t_q}(\psi, [ts'', te'']) \neq \top\big]\Big]\Big] \end{cases}$$

  where $\phi, \psi \in \Phi^-$. In other words, the temporal intersection of two formulae of Φ^- holds for the maximal intervals at which both formulae hold.

- $$\Lambda^-_{t_q}(\phi \ \mathsf{filter}_{\{<,\geq,=\}} \ n, [ts, te]) = \begin{cases} \top & \Lambda^-_{t_q}(\phi, [ts, te]) = \top \wedge te - ts \ \{<, \geq, =\} \ n \\ \bot & \text{otherwise.} \\ ? & \text{if } \mathsf{filter}_< : \Lambda^-_{t_q}(\phi, [ts, te]) = \ ? \\ & \text{if } \mathsf{filter}_{\{\geq,=\}} : \Lambda^-_{t_q}(\phi, [ts, te]) = \ ? \wedge te - ts \ \geq n \end{cases}$$ Consequently a filter formula ϕ $\mathsf{filter}_{\{<,\geq,=\}}$ n holds true at an interval $[ts, te]$ if the interval satisfies the duration constraint and ϕ is true at $[ts, te]$. A filter formula ϕ $\mathsf{filter}_{\{<,\geq,=\}}$ n holds unknown at an interval $[ts, te]$ if ϕ is unknown at $[ts, te]$ and exists a subinterval of $[ts, te]$ or $[ts, te]$ itself, that can satisfy the duration constraints.

- $$\Lambda^-_{t_q}(\phi \ \mathsf{filter}_{\{<,\geq,=\}} \ n, [ts, tq+]) = \begin{cases} \top & \text{if } \mathsf{filter}_\geq : \Lambda^-_{t_q}(\phi, [ts, tq+]) = \top \wedge tq - ts \ \geq \ n \\ \bot & \text{otherwise.} \\ ? & \text{if } \mathsf{filter}_{\{<,\geq,=\}} : \Lambda^-_{t_q}(\phi, [ts, tq+]) = \ ? \\ & \text{if } \mathsf{filter}_< : \Lambda^-_{t_q}(\phi, [ts, tq+]) = \top \wedge ts - tq \ < n \\ & \text{if } \mathsf{filter}_{\{\geq,=\}} : \Lambda^-_{t_q}(\phi, [ts, tq+]) = \top \wedge ts - tq \ \leq n \end{cases}$$ Here, a filter formula ϕ $\mathsf{filter}_{\{<,\geq,=\}}$ n can hold true only in the case of $\geq$ duration constraint. The filter formula will hold unknown for an interval $[ts, tq+]$ if ϕ is unknown at $[ts, tq+]$, or if the duration constraint cannot be confirmed yet—the extension of the interval may lead to the satisfaction or non-satisfaction of the duration constraints.

1.3 Formulae of $\phi \in \Phi^=$

The semantics for formulae $\phi \in \Phi^=$ are given by means of the valuation function $\Lambda^=_{t_q} : \Phi^= \times I \rightarrow \{\top, \bot, ?\}$. For simplicity we omit the cases of intervals of the form $[ts, tq+]$ as they are defined in similar manner. In what follows, we will use point intervals to refer to instants, we will use the valuation function Λ_{t_q} to denote that either of $\Lambda^{\bullet/-/=}_{t_q}$ can be used and the valuation function $\Lambda^\sim_{t_q}$ to denote that either of $\Lambda^{-/=}_{t_q}$ can be used. Moreover in order to avoid much lengthier semantics we assume that if $\Lambda^=_t(\phi, [ts, te]) = \top$ and $\Lambda^=_t(\phi, [ts, te]) =?$, then $\Lambda^=_t(\phi, [ts, te]) = \top$:

- $$\Lambda^=_{t_q}(P(a_1, ..., a_n), [ts, te]) = \begin{cases} \top & [ts, te] \in V^=(P(a_1, ..., a_n)) \\ \bot & [ts, te] \notin V^=(P(a_1, ..., a_n)) \\ ? & \text{never.} \end{cases}$$

  where P is n-ary dynamic temporal phenomenon predicate symbol.

- $$\Lambda^{=}_{t_q}(\phi \textsf{ before } \psi, [ts, te]) = \begin{cases} \top & \exists te'.\exists_{>te'} ts'.\Big[\Lambda_{t_q}(\phi, [ts, te']) = \top \wedge \Lambda_{t_q}(\psi, [ts', te]) = \top \\ & \quad \wedge \forall^{ts'}_{>te'} te''.\forall^{\le te'} ts''.\Lambda_{t_q}(\phi, [ts'', te'']) = \bot \\ & \quad \wedge \forall^{<ts'}_{>te'} ts''.\forall_{>ts''} te'' \Lambda_{t_q}(\psi, [ts'', te'']) = \bot \\ & \quad \wedge \forall^{<ts'}_{>te'} t.\nexists_{\le t} ts'' \nexists_{\ge t} te'' \big[\Lambda_{t_q}(\psi, [ts'', te'']) =? \\ & \qquad \vee \Lambda_{t_q}(\phi, [ts'', te'']) =?\big]\Big] \\ \bot & \text{otherwise.} \\ ? & \exists te'.\exists_{>te'} ts'.\Big[\Lambda_{t_q}(\phi, [ts, te']) = \top \wedge \Lambda_{t_q}(\phi, [ts', te]) = \top \\ & \quad \wedge \forall^{<ts'}_{>te'} te''.\forall^{\le te'} ts''.\Lambda_{t_q}(\phi, [ts'', te'']) = \bot \\ & \quad \wedge \forall^{<ts'}_{>te'} ts''.\forall_{>ts''} te''.\Lambda_{t_q}(\psi, [ts'', te'']) = \bot \\ & \quad \wedge \exists^{<ts'}_{>te'} t.\exists_{\le t} ts''.\exists_{\ge t} te''.\big[\Lambda_{t_q}(\psi, [ts'', te'']) =? \\ & \qquad \vee \Lambda_{t_q}(\phi, [ts'', te'']) =?\big]\Big] \\ & \vee \exists te'.\exists_{>te'} ts'.\Big[\Lambda_{t_q}(\phi, [ts, te']) =? \wedge \Lambda_{t_q}(\psi, [ts', te]) = \top \\ & \quad \wedge \forall^{<ts'}_{>te'} te''.\forall^{\le te'} ts''.\Lambda_{t_q}(\phi, [ts'', te'']) \neq \top \\ & \quad \wedge \forall^{<ts'}_{>te'} ts''.\forall_{>ts''} te''.\Lambda_{t_q}(\psi, [ts'', te'']) \neq \top\Big] \\ & \vee \exists te'.\exists_{>te'} ts'.\Big[\Lambda_{t_q}(\phi, [ts, te']) \neq \bot \wedge \Lambda_{t_q}(\psi, [ts', te]) =? \\ & \quad \wedge \forall^{<ts'}_{>te'} te''.\forall^{\le te'} ts''.\Lambda_{t_q}(\phi, [ts'', te'']) \neq \top \\ & \quad \wedge \forall^{<ts'}_{>te'} ts''.\forall_{>ts''} te''.\Lambda_{t_q}(\psi, [ts'', te'']) \neq \top \\ & \quad \wedge \exists_{>ts'} ts''.\exists_{\ge ts''} te''.\big[\Lambda_{t_q}(\phi, [ts'', te'']) = \top \\ & \qquad \wedge \forall^{<ts''}_{>ts'} ts'''.\forall_{>ts'''} te'''.\Lambda_{t_q}(\phi, [ts''', te''']) = \bot\big]\Big] \end{cases}$$

where $\phi, \psi \in \Phi^{\bullet} \cup \Phi^{-} \cup \Phi^{=}$. Clearly[2], before holds true if ϕ and ψ are true on two contiguous intervals/-points, and there is no interval/point starting or ending in the meantime at which ϕ or ψ has unknown status.

- $$\Lambda^{=}_{t_q}(\phi \textsf{ meets } \psi, [ts, te]) = \begin{cases} \top & \exists t' \Lambda^{\sim}_{t_q}(\phi, [ts, t']) = \top \wedge \Lambda^{\sim}_{t_q}(\phi, [t', te]) = \top \\ \bot & \text{otherwise.} \\ ? & \exists^{<te} te'.\exists^{\le te'} ts'.\big[\Lambda^{\sim}_{t_q}(\phi, [ts, te']) = \top \wedge \Lambda^{\sim}_{t_q}(\phi, [ts', te]) =?\big] \\ & \vee \exists te'.\exists^{\le te'}_{>ts} ts'.\big[\Lambda^{\sim}_{t_q}(\phi, [ts, te']) =? \wedge \Lambda^{\sim}_{t_q}(\phi, [ts', te]) = \top\big] \\ & \vee \exists te'.\exists^{\le te'} ts'.\exists_{>ts} te''.\big[\Lambda^{\sim}_{t_q}(\phi, [ts, te']) =? \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\phi, [ts', te'']) =? \wedge\ te = min(te', te'')\big] \end{cases}$$

where $\phi, \psi \in \Phi^{-} \cup \Phi^{=}$. Therefore ϕ meets ψ holds true at an interval $[ts, te]$, if ϕ and ψ hold true at the interval $[ts, t]$ and $[t, te]$ respectively. ϕ meets ψ holds unknown at an interval $[ts, te]$ if ϕ and ψ hold true and unknown (or unknown and true) at the intervals $[ts, te']$ and $[ts', te]$ respectively. Moreover, ϕ meets ψ holds unknown for an intervals $[ts, te]$ if ϕ and ψ hold unknown for intervals $[ts, te']$ and $[ts', te'']$ and $te = min(te', te'')$. Recall that if a formula holds unknown at an interval $[ts, te]$, it can possibly be true at any subinterval of $[ts, te]$.

- $$\Lambda^{=}_{t_q}(\phi \textsf{ overlaps } \psi, [ts, te]) = \begin{cases} \top & \exists_{>ts} ts'.\exists^{<te}_{>ts} te'.\big[\Lambda^{\sim}_{t_q}(\phi, [ts, te']) = \top \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\psi, [ts', te]) = \top\big] \\ \bot & \text{otherwise.} \\ ? & \exists^{<te} te'.\exists^{\le te'} ts'.\big[\Lambda^{\sim}_{t_q}(\phi, [ts, te']) = \top \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\psi, [ts', te]) =?\big] \\ & \vee \exists te'.\exists^{<te'}_{>ts} ts'.\big[\Lambda^{\sim}_{t_q}(\phi, [ts, te']) =? \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\phi, [ts', te]) = \top\big] \\ & \vee \exists te'.\exists^{<te'} ts'.\exists_{>ts} te''.\big[\Lambda^{\sim}_{t_q}(\phi, [ts, te']) =? \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\phi, [ts', te'']) =? \wedge\ te = min(te', te'')\big] \end{cases}$$

where $\phi, \psi \in \Phi^{-} \cup \Phi^{=}$. Therefore, ϕ overlaps ψ holds true if the pair of the intervals that satisfy the relation constraints correspond to true valuations. If any of the participating intervals is unknown (or both) and there are subintervals that can satisfy the relation constraints then the formula holds for an interval with an unknown truth value.

[2] The three valued semantics of ϕ before ψ follow the same intuition as the semantics of $\phi \looparrowright \psi$

- $$\Lambda^{=}_{t_q}(\phi \text{ contains } \psi, [ts,te]) = \begin{cases} \top & \exists_{>ts} ts'.\exists^{<te} te'.\big[\Lambda^{\sim}_{t_q}(\phi,[ts,te]) = \top \\ & \quad \wedge \Lambda_{t_q}(\psi,[ts',te']) = \top\big] \\ \bot & \text{otherwise.} \\ ? & \exists^{\leq te}_{>ts} t.\exists^{\leq t} ts'.\exists_{\geq t} te'.\ \big[\Lambda^{\sim}_{t_q}(\phi,[ts,te]) = \top \\ & \quad \wedge \Lambda_{t_q}(\psi,[ts',te']) = ?\big] \\ & \vee \exists_{>ts} ts'.\exists^{<te} te'.\ \big[\Lambda^{\sim}_{t_q}(\phi,[ts,te]) = ? \\ & \quad \wedge \Lambda_{t_q}(\psi,[ts',te']) = \top\big] \\ & \vee \exists^{<te}_{>ts} t.\exists^{<t} ts' \exists_{>t} te'.\ \big[\Lambda^{\sim}_{t_q}(\phi,[ts,te]) = ? \\ & \quad \wedge \Lambda_{t_q}(\psi,[ts',te']) = ? \wedge\ te = min(te',te'')\big] \end{cases}$$

  where $\phi \in \Phi^{\bullet} \cup \Phi^{-} \cup \Phi^{=}$ and $\psi \in \Phi^{-} \cup \Phi^{=}$. Similar to the semantics of overlaps, ϕ contains ψ holds true for an interval when the participating intervals satisfy the relation constraints and they correspond to true valuations. The valuation status is unknown when any of the participating intervals has unknown status and there are subintervals that can satisfy the constraints of the contains relation in which case the limits of the resulting interval are adjusted accordingly.

- $$\Lambda^{=}_{t_q}(\phi \text{ finishes } \psi, [ts,te]) = \begin{cases} \top & \exists_{>ts} ts'.\big[\Lambda_{t_q}(\phi,[ts',te]) = \top \wedge \Lambda^{\sim}_{t_q}(\psi,[ts,te]) = \top\big] \\ \bot & \text{otherwise.} \\ ? & \exists_{\geq te} te'.\exists^{<te} ts'.\big[ts' < te' \wedge \Lambda^{\sim}_{t_q}(\phi,[ts',te']) = ? \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\phi,[ts,te]) = \top\big] \\ & \vee \exists t.\big[t = te \wedge \Lambda^{\bullet}_{t_q}(\phi,[t,t]) = ? \wedge\ \Lambda^{\sim}_{t_q}(\psi,[ts,te]) = \top\big] \\ & \vee \exists te'.\exists^{<te'} ts'.\exists_{\geq te'} te''.\big[\Lambda^{\sim}_{t_q}(\phi,[ts',te'']) = \top \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\psi,[ts,te']) = ? \wedge te = min(te',te'')\big] \\ & \vee \exists_{\geq te} t'.\big[\Lambda^{\bullet}_{t_q}(\phi,[te,te]) = \top \wedge \Lambda^{\sim}_{t_q}(\psi,[ts,te']) = ?\big] \\ & \vee \exists te'.\exists_{\leq te'} ts'.\exists_{\geq ts} te''.\big[ts' < te' \wedge \Lambda_{t_q}(\phi,[ts',te'']) = ? \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\psi,[ts,te']) = \top \wedge te = min(te'',te')\big] \end{cases}$$

  where $\phi \in \Phi^{\bullet} \cup \Phi^{-} \cup \Phi^{=}$ and $\psi \in \Phi^{-} \cup \Phi^{=}$. The semantics of ϕ finishes ψ follow the same intuition as contains and overlaps. For example, in the case of an unknown valuation status and the following intervals $[1,10],[2,9]$ where ϕ and ψ are unknown respectively, ϕ finishes ψ will hold for the interval $[2,9]$ with unknown status, as there are subintervals of $[1,10]$ (e.g., $[3,9]$) that can satisfy the constraints of the relation finishes.

- $$\Lambda^{=}_{t_q}(\phi \text{ starts } \psi, [ts,te]) = \begin{cases} \top & \exists_{<te} te'.\Lambda_{t_q}(\phi,[ts,te']) = \top \wedge \Lambda^{\sim}_{t_q}(\psi,[ts,te]) = \top \\ \bot & \text{otherwise.} \\ ? & \exists_{\geq ts} te'.\exists^{\leq ts} ts'.\big[ts' < te' \wedge \Lambda^{\sim}_{t_q}(\phi,[ts',te']) = ? \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\psi,[ts,te]) = \top\big] \\ & \vee \exists t.\big[t = ts \wedge \Lambda^{-}_{t_q}(\phi,[t,t]) = ? \\ & \quad \wedge \Lambda^{\bullet}_{t_q}(\phi,[ts,te]) = \top\big] \\ & \vee \exists te'.\exists_{<te} ts''.\exists^{\leq ts''} ts'.\big[\Lambda^{\sim}_{t_q}(\phi,[ts',te'']) = \top \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\phi,[ts'',te']) = ? \wedge ts = max(ts',ts'')\big] \\ & \vee \exists_{\leq ts} ts'.\ \big[\Lambda^{\bullet}_{t_q}(\phi,[te,te]) = \top \wedge \Lambda^{\sim}_{t_q}(\phi,[ts',te]) = ?\big] \\ & \vee \exists ts'.\exists_{\geq} ts'.\exists^{\leq te} ts''.\big[\Lambda_{t_q}(\phi,[ts'',te']) = ? \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\phi,[ts',te]) = ? \wedge ts = max(ts'',ts')\big] \end{cases}$$

  where $\phi \in \Phi^{\bullet} \cup \Phi^{-} \cup \Phi^{=}$ and $\psi \in \Phi^{-} \cup \Phi^{=}$. The semantics of ϕ starts ψ are defined in a similar manner as ϕ finishes ψ.

- $$\Lambda^{=}_{t_q}(\phi \text{ equals } \psi, [ts,te]) = \begin{cases} \top & \Lambda^{\sim}_{t_q}(\phi,[ts,te]) = \top \wedge \Lambda^{\sim}_{t_q}(\psi,[ts,te]) = \top \\ \bot & \text{otherwise.} \\ ? & \exists^{\leq ts} ts'.\exists_{\geq te'} te.\big[\Lambda^{\sim}_{t_q}(\psi,[ts,te]) = \top \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\psi,[ts',te']) = ?\big] \\ & \vee \exists^{\leq ts} ts'.\exists_{\geq te'} te.\big[\Lambda^{\sim}_{t_q}(\psi,[ts',te']) = ? \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\psi,[ts,te]) = \top\big] \\ & \vee \exists ts'.\exists te'.\exists^{\leq te'} ts''.\exists^{\geq ts'} te''.\big[\Lambda^{\sim}_{t_q}(\psi,[ts',te']) = ? \\ & \quad \wedge \Lambda^{\sim}_{t_q}(\psi,[ts'',te'']) = ? \wedge ts = max[ts',ts''] \\ & \quad \wedge te = min(te'',te')\big] \end{cases}$$

  where $\phi, \psi \in \Phi^{-} \cup \Phi^{=}$.

  Similar to the above cases, ϕ equals ψ holds unknown if any of the participating intervals are produced from valuations with unknown status, and there are subintervals that can satisfy the constraints of the relation equals.

References

[1] Kleene, S. C. *Introduction to Metamathematics*. Princeton, NJ, USA: North Holland, 1952.

[2] Pitsikalis, M., Lisitsa, A., and Luo, S. Representation and processing of instantaneous and durative temporal phenomena. In *Logic-Based Program Synthesis and Transformation* (Cham, 2022), E. De Angelis and W. Vanhoof, Eds., Springer International Publishing, pp. 135–156.